The S-matrix for a 4-port directional coupler is

$$\mathbf{S} = \begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{bmatrix} \quad [\ 1\ ]$$

If we assume that all four ports are matched this gives $S_{1,1} = S_{2,2} = S_{3,3} = S_{4,4} = 0$. If we add the stipulation (as is the case for a directional coupler) that ports 1 and 2 are isolated and ports 3 and 4 are isolated, the additional zeros are entered into the S-matrix: $S_{1,2} = S_{2,1} = S_{3,4} = S_{4,3} = 0$. Furthermore, due to reciprocity of the passive network we can readily assume that $S_{i,j} = S_{j,i}$ for all i, j ranging from 1 through 4. Incorporating all these conditions into [1] above gives the S-matrix for an ideal directional coupler.

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & S_{1,3} & S_{1,4} \\ 0 & 0 & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & 0 & 0 \\ S_{4,1} & S_{4,2} & 0 & 0 \end{bmatrix} \quad [\ 2\ ]$$

From the unitary property of the S-matrix { $\left(S^T\right)^* S = I$ } the following identities can be written:

$$\begin{matrix} S_{1,3}{}^* S_{2,3} + S_{1,4}{}^* S_{2,4} = 0 \\ S_{1,3}{}^* S_{1,4} + S_{2,3}{}^* S_{2,4} = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \left|S_{1,3}\right|\left|S_{2,3}\right| = \left|S_{1,4}\right|\left|S_{2,4}\right| \\ \left|S_{1,3}\right|\left|S_{1,4}\right| = \left|S_{2,3}\right|\left|S_{2,4}\right| \end{matrix} \quad [\ 3\ ]$$

Dividing the second pair of equations by each other in [3] reveals that $\left|S_{2,3}\right| = \left|S_{1,4}\right|$. One possible solution which satisfies the constraints imposed by the unitary property of the S-matrix is in [4] below.

$$\begin{matrix} S_{1,3} = S_{2,4} = \alpha \\ S_{2,3} = S_{1,4} = j\beta \end{matrix} \quad [\ 4\ ]$$

The scattering matrix for the 4-port directional coupler is then given in [5].

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & \alpha & j\beta \\ 0 & 0 & j\beta & \alpha \\ \alpha & j\beta & 0 & 0 \\ j\beta & \alpha & 0 & 0 \end{bmatrix} \quad [\ 5\ ]$$

If the fourth port of the directional coupler is terminated in a matched load, the S-matrix becomes a 3 x 3 matrix with the fourth column and fourth row eliminated from the matrix in [5]. The S-matrix for a power combiner/divider becomes that in [6] below.

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & j\beta \\ \alpha & j\beta & 0 \end{bmatrix} \quad [\ 6\ ]$$

Special Case - Wilkinson Divider/Combiner

The Wilkinson power divider/combiner for equal power combining/dividing is shown in Figure 1. The output signals at ports 2 and 3 are equal phase, equal amplitude in nature for this arrangement.

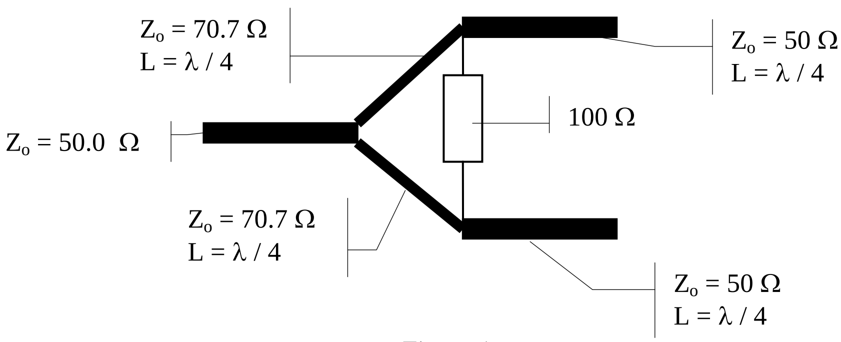

Figure 1
Equal-Phase, Equal-Amplitude Wilkinson Divider/Combiner

It is possible to form Wilkinson dividers/combiners which have unequal amplitude characteristics. A necessary condition is that the two coupled ports are assumed to have identical phase conditions which, when using balanced amplifiers, will vary somewhat and affect the overall efficiency of the combiners/divider.

The governing relationships used in the design of an unequal amplitude Wilkinson are listed below in Table I.

$K^2 = \dfrac{power\ at\ port\ 1}{power\ at\ port\ 2} = \dfrac{\alpha^2}{\beta^2} > 1$
$Z_{01} = Z_o \left(\dfrac{K}{1+K^2} \right)^{1/4}$
$Z_{02} = Z_o\ K^{3/4} \left(1+K^2 \right)^{1/4}$
$Z_{03} = Z_o \left(\dfrac{1+K^2}{K^5} \right)^{1/4}$
$Z_{04} = Z_o \sqrt{K}$
$Z_{05} = \dfrac{Z_o}{\sqrt{K}}$
$R = Z_o \left(\dfrac{1+K^2}{K} \right)$

Table I
Unequal Power Division Wilkinson

The S-matrix relationships for this Wilkinson are shown below in [7]. Both α and β are real numbers, indicative of no phase difference between the two of them. Writing out the expression for $|\ b_3\ |^2$, which is the power delivered to the sum-port is given in [8].

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha & \beta & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \qquad [\ 7\ ]$$

$$\left| b_3 \right|^2 = \left| \alpha a_1 + \beta a_2 \right|^2 \qquad [\ 8\ ]$$

The coefficients α and β determine what the power division ratio is between the two ports, 1 and 2.

Example: We want a 2 : 1 power division/combining ratio between the two input ports and the output port. Equation [8] can be written in terms of the single variable a_2:

$$\left| b_3 \right|^2 = \Big| \alpha \left| a_2 \right| \sqrt{2} + \beta \left| a_2 \right| \Big|^2$$

Performing the required algebra gives the following:

$$\begin{aligned} \left| b_3 \right|^2 &= \left| a_1 \right|^2 + \left| a_2 \right|^2 = 3 \left| a_2 \right|^2 \\ \left(\alpha \sqrt{2} + \beta \right)^2 &\equiv 3 \end{aligned} \qquad [\ 9\ ]$$

Using the results in [9] and the required conditions imposed by $\alpha^2 + \beta^2 = 1$, the following factors give the 2 : 1 power division needed.

$$\alpha = \frac{\sqrt{6}}{3} \qquad \beta = \frac{1}{\sqrt{3}} \qquad [\ 10\ ]$$

It is interesting to note the following:

$$20\log_{10}(\alpha)=20\log_{10}\left(\frac{\sqrt{6}}{3}\right)=-1.76\ dB \qquad 20\log_{10}(\beta)=20\log_{10}\left(\frac{1}{\sqrt{3}}\right)=-4.77\ dB$$

The unequal combining Wilkinson is illustrated below in Figure 2 with the accompanying details derived from the expressions in Table I.

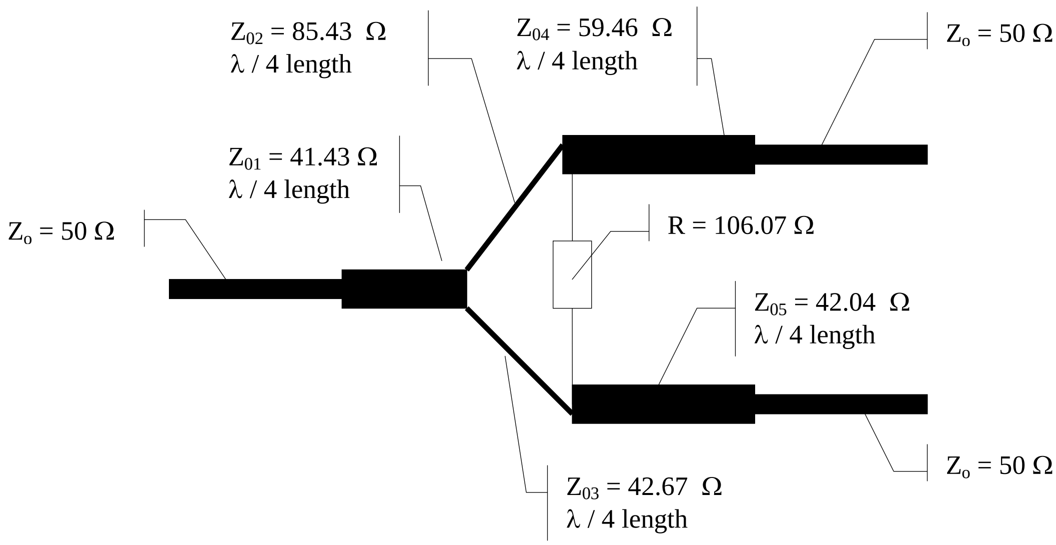

Figure 2
Unequal-Division Wilkinson: Specific 2:1 Power Division Ratio

