Instantaneous power: $p = \frac{dw}{dt}$ time-rate of change of energy

The power is measured in watts when the voltage is in volts and the current in amperes.

$$v = V_m \cos(\omega t + \theta_v) \qquad i = I_m \cos(\omega t + \theta_i) \qquad [11.2\text{-}3]$$

Power engineers have traditionally set the "zero time" to correspond to the instant of time when the current is passing through a positive maximum. This reference system requires that we shift both the voltage and current by θ_i. This causes the expressions above to become:

$$v = V_m \cos(\omega t + \theta_v - \theta_i) \qquad i = I_m \cos(\omega t) \qquad [11.4\text{-}5]$$

Substituting [11.4] and [11.5] into [11-1] for power gives:

$$p = vi = V_m I_m \cos(\omega t + \theta_v - \theta_i)\cos(\omega t) \qquad [11.6]$$

11.2 Real & Reactive Power

$$P = \frac{1}{T}\int_{t_o}^{t_o+T} p\, dt \qquad [11.7]$$

We use the trigonometric identify to simplify the expression of [11-6]:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

Letting $\alpha = wt + \theta_v - \theta_i$ and $\beta = \omega t$ we can write [11.6] as

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i) \qquad [11\text{-}8]$$

Now using another trigonometric identity

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

to expand the second term on the right-hand side of [11-8]

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t) \qquad [11\text{-}9]$$

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) \qquad [11\text{-}10]$$

1. The average power of p is given by the first term on the right hand side. The integral of the two trig functions over one period is identically zero.
2. The frequency of the instantaneous power is twice the frequency of the voltage or the current.
3. If the circuit is purely resistive, the voltage and current will be in phase. This means that $\theta_v = \theta_i$ and [11-9] reduces to $p = P + P\cos(2\omega t)$ [11-11]
4. If the circuit is entirely inductive, the voltage and current will be out of phase 90^o. In this case the instantaneous power reduces to $p = -\frac{V_m I_m}{2}\sin(2\omega t)$ [11-12]

5. For a capacitive circuit $$p = \frac{V_m I_m}{2}\sin(2\omega t) \qquad [11\text{-}13]$$

The power associated with purely inductive or capacitive circuits is referred to as reactive power, Q.

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) \qquad [11\text{-}14]$$

$$p = P + P\cos(2\omega t) - Q\sin(2\omega t) \qquad [11\text{-}15]$$

Power Factor: $$pf = \cos(\theta_v - \theta_i) \qquad [11\text{-}16]$$

11.5 Power Calculations

Complex power is the complex sum of the average real power and the reactive power: $S = P + jQ$

Complex power is in terms of volt-amps; watts for average real power, and vars for reactive power. The magnitude of the complex power is termed apparent power.

$$\begin{aligned} S &= \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i) \\ &= \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right] \qquad [11\text{-}30] \\ &= \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i) \end{aligned}$$

$$S = \frac{1}{2} V_m I_m e^{j(\theta_v - \theta_i)} = V_{eff} I_{eff} \angle(\theta_v - \theta_i)$$

$$now\ V_m I_m e^{j(\theta_v - \theta_i)} = V_m e^{j\theta} I_m e^{-j\theta_i} = V_m e^{j\theta}\left(I_m e^{j\theta_i}\right)^* = \mathbf{V}\mathbf{I}^* \qquad [11\text{-}32,33]$$

$$S = \frac{1}{2}\mathbf{V}\,\mathbf{I}^* = \mathbf{V}_{eff}\mathbf{I}_{eff}^* = P + jQ$$

The first variation of [11-33] is to replace the voltage by the product of the current times the impedance.

$$\mathbf{V}_{eff} = Z\,\mathbf{I}_{eff} \qquad [11\text{-}34]$$

Substituting:

$$S = Z\,\mathbf{I}_{eff}\mathbf{I}_{eff}^* = \left|\mathbf{I}_{eff}\right|^2 Z = \left|\mathbf{I}_{eff}\right|^2 (R + jX)$$

$$S = \left|\mathbf{I}_{eff}\right|^2 R + j\left|\mathbf{I}_{eff}\right|^2 X = P + jQ$$

$$P = \left|\mathbf{I}_{eff}\right|^2 R = \frac{1}{2} I_m^2 R \qquad Q = \left|\mathbf{I}_{eff}\right|^2 X = \frac{1}{2} I_m^2 X \qquad [11\text{-}36,37]$$

A second useful variation of [11-33] is by replacing the current by the voltage divided by the impedance.

$$S = \mathbf{V}_{eff}\left(\frac{\mathbf{V}_{eff}}{Z}\right)^* = \frac{\left|\mathbf{V}_{eff}\right|^2}{Z^*} = P + jQ \qquad [11\text{-}38]$$

11.8 Maximum Power Transfer

For a network operating in the sinusoidal steady-state, maximum average power is delivered to a load when the load impedance is the conjugate of the Thevenin impedance of the network as viewed from the terminals of the load.

$$Z_L = Z_{Th}^* \qquad [11\text{-}41]$$

This can be derived beginning with the following:

$$Z_{Th} = R_{Th} + jX_{Th} \qquad Z_L = R_L + jX_L \qquad [11\text{-}42,43]$$

Since we are making an average power calculation we assume that the amplitude of the Thevenin voltage is expressed in terms of its rms value.

Therefore, the rms value of the load current **I** is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}\angle 0^o}{\left(R_{Th}+R_L\right)+j\left(X_{Th}+X_L\right)} \qquad [11\text{-}44]$$

The average power delivered to the load is $P = \left|\mathbf{I}\right|^2 R_L$ [11-45]

Substituting [11-44] into [11-45] gives for P:

$$P = \frac{\left|\mathbf{V}_{Th}\right|^2 R_L}{\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2} \qquad [11\text{-}46]$$

In working with [11-46] it is important to remember that $\mathbf{V}_{Th}$, R_{Th}, and X_{Th} are all fixed quantities, whereas R_L and X_L are independent variables. Therefore, to maximize P, we must find the values of R_L and X_L, where $\partial P/\partial R_L$ and $\partial P/\partial X_L$ are both zero.

$$\frac{\partial P}{\partial R_L} = \frac{\left[\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2\right]\left|V_{Th}\right|^2 - 2\left(R_{Th}+R_L\right)\left|V_{Th}\right|^2 R_L}{\left[\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2\right]} = 0 \qquad [11\text{-}47]$$

$$\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2 = 2R_L\left(R_{Th}+R_L\right)$$

$$\frac{\partial P}{\partial X_L} = \frac{\left[\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2\right]\times 0 - \left|V_{Th}\right|^2 R_L 2\left(X_{Th}+X_L\right)}{\left[\left(R_{Th}+R_L\right)^2+\left(X_{Th}+X_L\right)^2\right]^2} = 0 \qquad [11\text{-}48]$$

From [11-48] we see that $\partial P/\partial X_L$ is identically zero when $X_L = -X_{Th}$. Making this substitution in [11-47] and simplifying gives:

$$\left(R_{Th}+R_L\right)^2 = 2R_L\left(R_{Th}+R_L\right)$$

$$R_{Th}^2 + 2R_{Th}R_L + R_L^2 - 2R_LR_{Th} - 2R_L^2 = 0$$

$$R_{Th}^2 = R_L^2 \qquad R_L = R_{Th}$$

We can also see from $\partial P/\partial R_L$ and [11-48] that

$$R_L = \sqrt{R_{Th}^2 + \left(X_L + X_{Th}\right)^2} \qquad [11\text{-}50]$$

The maximum average power that can be delivered to Z_L when it is set equal to the conjugate of Z_{Th} is calculated directly.

When $Z_L = Z_{Th}^*$ the rms load current is $\mathbf{V}_{Th}/2R_L$ and the maximum average power delivered to the load is (maximum average indicates no reactive contributions)

$$P_{\max} = \frac{|\mathbf{V}_{Th}|^2 R_L}{4R_L^2} = \frac{1}{4}\frac{|\mathbf{V}_{Th}|^2}{R_L} \qquad [11\text{-}51]$$

If the Thevenin voltage is expressed in terms of its maximum amplitude rather than its rms amplitude, [11-51] becomes

$$P_{\max} = \frac{1}{8}\frac{{V_m}^2}{R_L} \qquad [11\text{-}52]$$

The maximum average power can be delivered to Z_L only if Z_L can be set equal to the conjugate of Z_{Th}. There are situations where this is not possible.

R_L and X_L may be restricted to a limited range of values. In this situation, the optimum condition for R_L and X_L is to adjust X_L as near $-X_{Th}$ as possible and then adjust R_L as close to $\sqrt{R_{Th}^2 + \left(X_L + X_{Th}\right)^2}$ as possible.

A second type of restriction is when the magnitude of Z_L can be varied but not its phase. Under this restriction the greatest amount of power is transferred to the load when the magnitude of Z_L is set equal to the magnitude of Z_{Th}.

Impedance Condition	*Real Part*	*Reactive Part*
Perfect Match	$R_L = R_{Th}$	$X_L = -X_{Th}$
Best Match Possible	$R_L = \sqrt{R_{Th}^2 + \left(X_L + X_{Th}\right)^2}$	Set X_L as close as possible to $-X_{Th}$
Impedance Magnitude	$\|Z_L\| = \|Z_{Th}\|$	$\|Z_L\| = \|Z_{Th}\|$

14.2 Resonance

For a parallel resonant circuit fed by a current source:

$$\mathbf{V}_o = \frac{\mathbf{I}_s}{Y_s} = \frac{\mathbf{I}_s}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} \qquad [14\text{-}1]$$

The amplitude of the voltage is

$$V_m = \frac{I_m}{\sqrt{\left(\frac{1}{R}\right)^2 + \left[\omega C - \frac{1}{\omega L}\right]^2}}$$

The phase angle is

$$\tan(\phi) = \left(\omega C - \frac{1}{\omega L}\right)R$$

The resonant frequency for the circuit is defined as the frequency that makes the impedance seen by the current source purely resistive.

$I := 1 \qquad R := 50 \quad L := 7.5 \cdot 10^{-6} \qquad C := 150 \cdot 10^{-12}$

$k := 1, 2 .. 100$

$\omega_k := 5 \cdot 10^5 \cdot k$

$Y_k := \frac{1}{R} + j \cdot \left[(\omega_k) \cdot C - \frac{1}{\omega_k \cdot L} \right] \qquad \theta_k := \text{atan2}\left[\left(\frac{1}{R} \right), \omega_k \cdot C - \frac{1}{\omega_k \cdot L} \right] \cdot \frac{180}{\pi}$